On Modular Extensions to Nim

Karan Sarkar

Mentor: Dr. Tanya Khovanova

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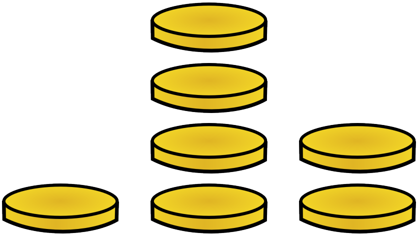
16 May 2015

An Instructive Example: Nim

The Rules

- Take at least one token from some chosen pile.
- Player who takes last token wins.



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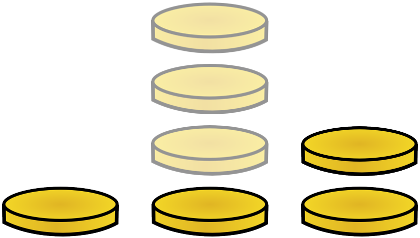

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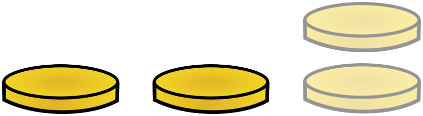

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Nim Positions

Position Notation

A position with piles of sizes $a_1, a_2, \ldots, a_n$ is denoted as the ordered n-tuple:

$$(a_1, a_2, \ldots, a_n).$$

Definition

A P-position is a position that guarantees a ***loss*** given optimal play

Definition

An N-position is a position that guarantees a ***win*** given optimal play

The Winning Strategy for Nim

Theorem (Bouton's Theorem)

The position $(a_1, a_2, \ldots, a_n)$ is a P-position in Nim if and only if

$$\bigoplus_{i=1}^{n} a_i = 0.$$

Definition (Bitwise XOR)

The $\oplus$ symbol denotes the bitwise XOR operation.

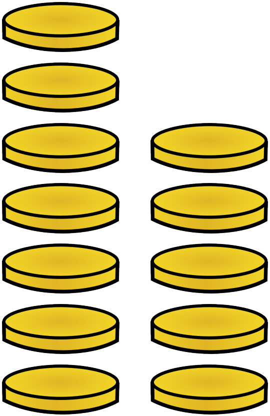
1. Write both numbers in binary.
2. Add without carrying over.

m-Modular Nim

The Rules

- Take at least one token from some chosen pile or km tokens total.
- Player who takes last token wins.

An Example: 3-Modular Nim with 2 Piles



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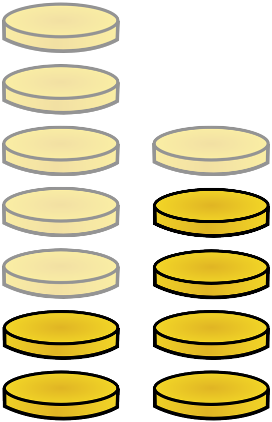

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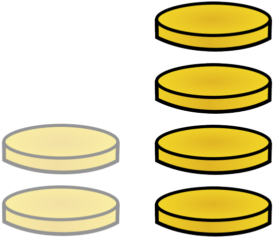

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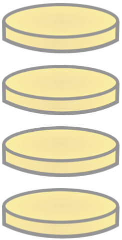

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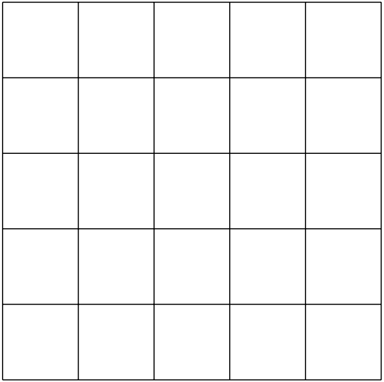

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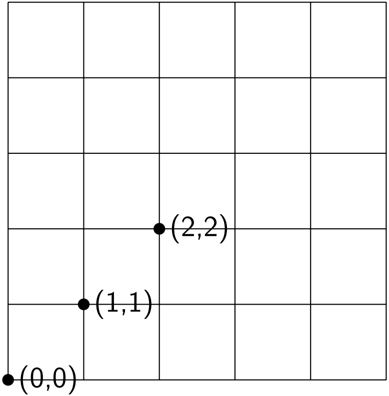

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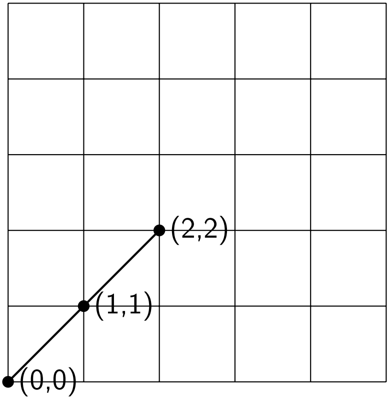

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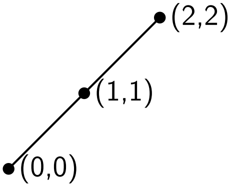
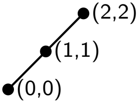


2 Heap m-Modular Nim for Odd m

Theorem

For odd m, a position of m-Modular Nim with 2 heaps is a P-position if and only it is of the form (i, i) for integers i where $0 \leq i < m$.

An Example: 6-Modular Nim with 2 Piles



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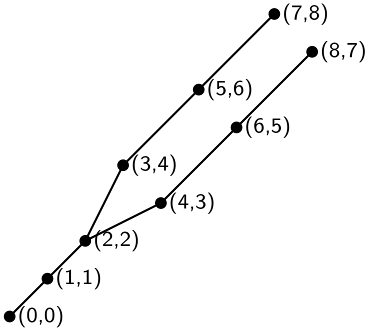

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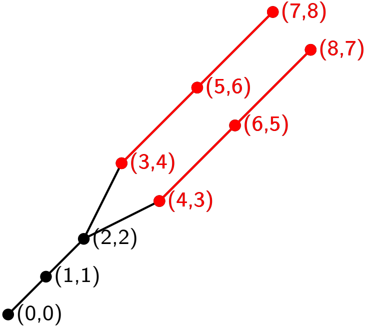
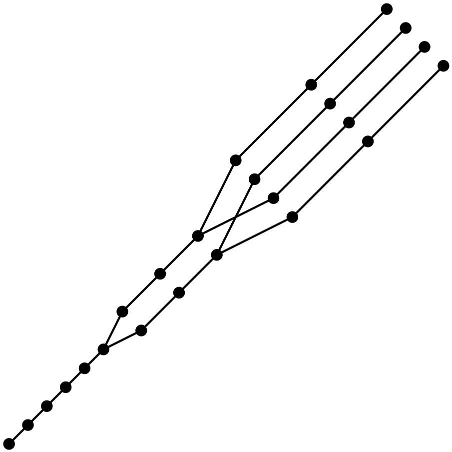


Another Example: 12-Modular Nim with 2 Piles



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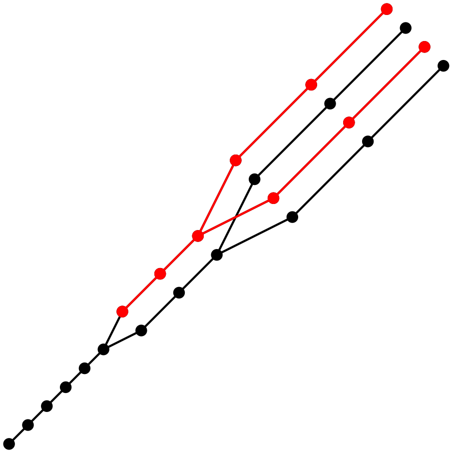

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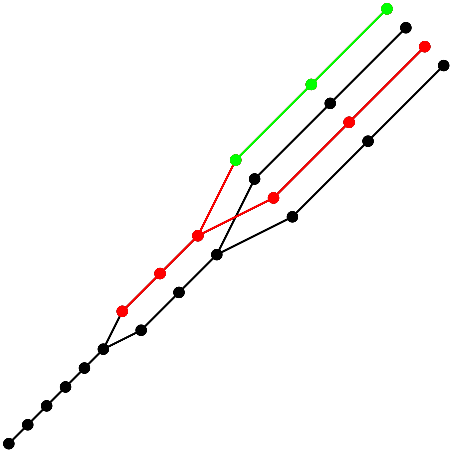


m-Modular Nim for 2 Heaps

Theorem

Let $m = 2^i \cdot k$ where k is odd. A position is a P-position if and only if it is of the form:

$$(2^{j-1} \cdot b + a, (k+1)2^{j-1} - 1 - a)$$

for all $0 \leq a < 2^{j-1}$, $k \leq b < 2k$ and $0 \leq j < i$.

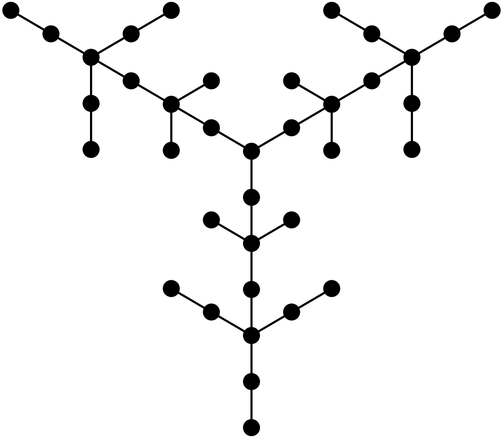
Corollary

Let $m = 2^i \cdot k$ where k is odd. There are

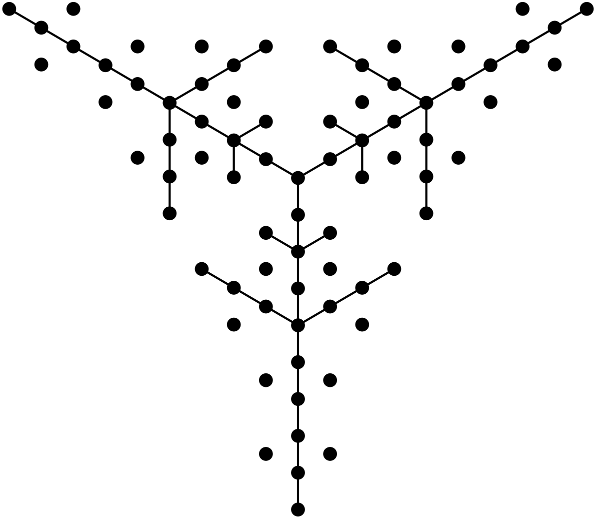
$$m\left(\frac{i}{2} + 1\right)$$

P-positions in m-Modular Nim with 2 heaps.

7-Modular Nim with 3 Heaps



A Snapshot of Nim with 3 Heaps



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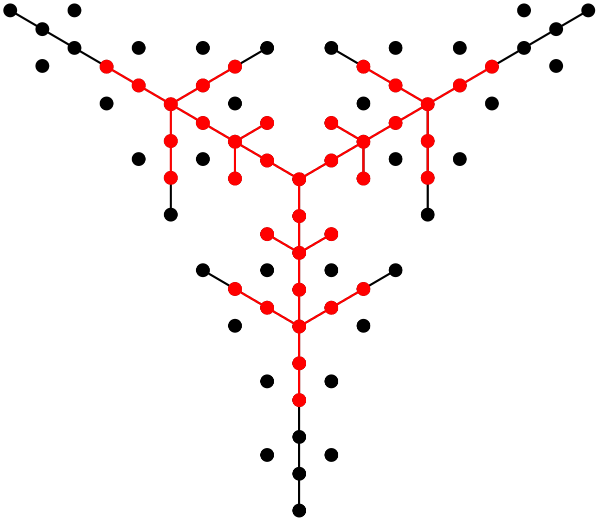

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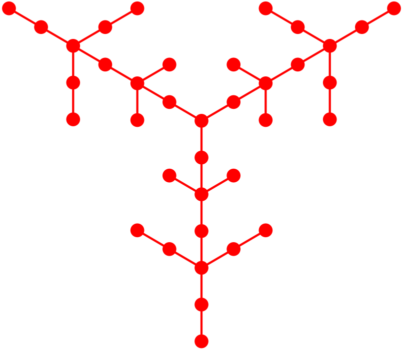

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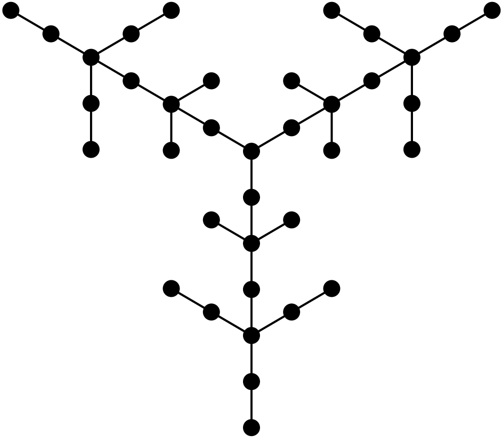


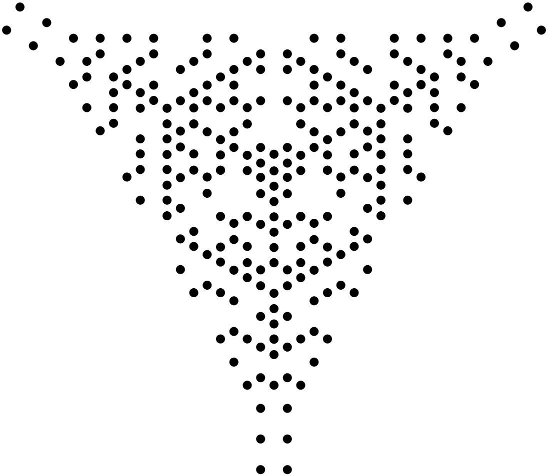
m-Modular Nim for odd m

Theorem

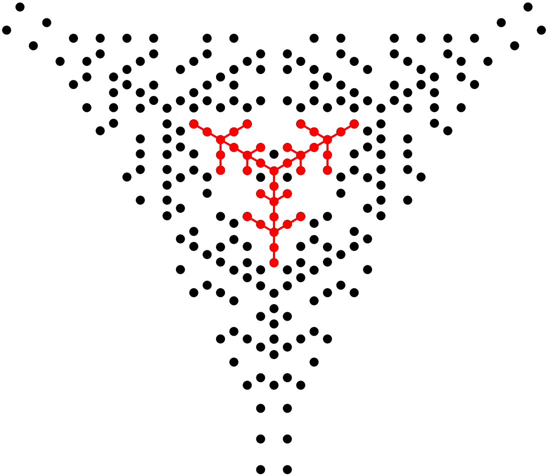
A position $(a_1, a_2, \ldots, a_n)$ is a P-position if and only if:

1. $\displaystyle\bigoplus_{i=1}^{n} a_i = 0.$
2. $\displaystyle\sum_{i=1}^{n} a_i < 2m.$

14-Modular Nim with 3 Heaps



14-Modular Nim with 3 Heaps



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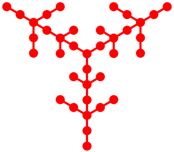

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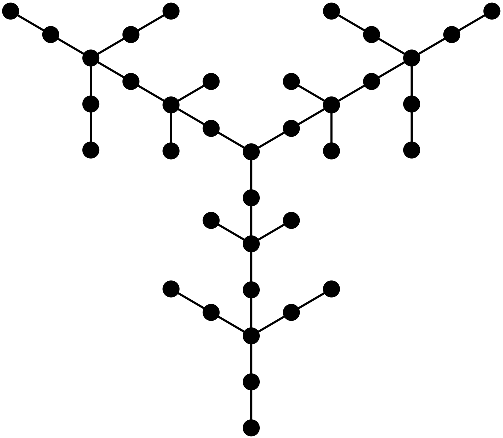

m-Modular Nim for Even m: A Partial Result

Theorem

If a position $(a_1, a_2, \ldots, a_n)$ is a P-position in m-Modular Nim for m odd, then it is a P-position in $2m$-Modular Nim.

Future Research

- What happens in Miseré Modular Nim?
- How do the P-positions for even m behave?
- What happens one can take away $km + r$ tokens for other values of r?
- What about other polynomials?

Acknowledgments

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- My parents: for their encouragement and transportation